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TOPICAL ISSUE

Robust Controller Design for Multivariable Systems under Nonstationary Parametric Variations and Bounded External Disturbances

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Abstract—This paper considers linear multivariable systems with physical parameters varying from their known nominal values in an arbitrary and nonstationary manner. The plant is subjected to polyharmonic external disturbances containing an arbitrary number of unknown frequencies with unknown amplitudes having a bounded sum. The problem is to design a controller that robustly stabilizes the closed loop system and ensures desired errors for the controlled variables of the plant with nominal parameters in the steady-state mode. The system equations of the original problem are represented in the (W, Λ, K)-form; for this form, the standard H_∞ optimization problem is stated and solved. The desired accuracy of the system is achieved by analytically assigning the weight matrix of the controlled variables. The controller design method is illustrated by an example of solving a well-known benchmark problem.

Keywords: linear multivariable systems, robust control, controller design, H_∞ optimization, bounded external disturbances, nonstationary parameters

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1. INTRODUCTION

Since the mid-1980s, the problem of robust stability analysis and robust stabilization has been receiving the close attention of researchers; see numerous references to the publications of this period, e.g., in [1–4]. These monographs presented in detail the known methods and approaches to solving the corresponding problems. Let us emphasize the following fundamental features of popular analysis and design methods of modern control theory:

—Well-known techniques, namely, H_2, H_∞, and l_1 optimization, μ-synthesis, and modal control (pole placement), may result in systems with very low robustness, i.e., unacceptably small stability margins (phase and gain margins) at the physical input or output of the plant.

—The controller's order may be much higher than that of the physical plant.

—Often only state-feedback controllers are designed, without loops by the measured physical output of the plant.

—The state equations (and the transfer matrix) are secondary descriptions, and their coefficients often have no physical meaning.

It is natural to consider equations in physical variables (based on the laws of mechanics and electrodynamics), as they have coefficients with clear physical interpretations (mass, moment of inertia, capacitance, inductance, resistance, etc.). The transition from the original equations in physical variables to state equations or transfer matrices mixes and multiplies the varying parameters. This strongly increases the conservatism of analysis and design results.

This paper deals with the plant's equations in physical variables. The approach proposed below is based on the (W, Λ)- and (W, Λ, K)-forms of equations introduced previously in [5–8]. The circle criterion of absolute stability [9] and the radius of stability margins [10] are adopted to show that the results of the cited works remain valid for the nonstationary variations of the plant's physical parameters from the nominal ones within the same bounds. This fact is crucial from both theoretical and engineering points of view.

In practice, real dynamical systems are subject to unmeasured external disturbances; in the mathematical theory of automatic control, they are considered bounded in some norm [11]. Therefore, the problem is to ensure admissible deviations of the plant's controlled variables from zero. Many studies have been devoted to the attenuation of external disturbances; for example, see [11, 12]. In this paper, unmeasured external disturbances are polyharmonic functions with an unknown (infinite) number of frequencies and unknown harmonic amplitudes whose sum is bounded by a given number, as in [10]. In the special case of multiple frequencies, these disturbances cover the entire class of physically possible disturbances in engineering practice: they are continuous and have piecewise continuous time derivatives and can be therefore represented by absolutely convergent Fourier series [10].

Thus, below we consider the problem of designing an output-feedback controller that robustly stabilizes the closed loop system under nonstationary variations of the plant's physical parameters from their nominal values (on the one hand) and ensures given deviations of the controlled variables from zero under the external disturbances of the class mentioned above (on the other hand). In a certain sense, this problem can be treated as the design of linear parameter-varying (LPV) systems [13–16], but, within this research line, the issues of given accuracy are not considered and the proof of stability involves Lyapunov's second method. Such an approach is less effective compared to the frequency criteria used below, due to the well-known difficulties in choosing an appropriate Lyapunov function. In addition, the structure of the (W, Λ, K)-form allows directly operating the loops of the varying parameters. As a result, the robust stabilization problem (controller design ensuring the required radius of stability margins in the loops of the varying physical parameters) can be solved within necessary and sufficient conditions for the radius of stability margins. This is the main advantage of the current robust controller design approach compared to all methods known in the literature.

The resulting problem is reduced to the standard H_∞ optimization one, and a given accuracy is achieved by assigning an appropriate diagonal weight matrix for the plant's variables; the elements of this matrix are assigned using analytically derived formulas, similar to [10]. Note that the controller's order does not exceed that of the original physical plant.

The proposed approach was implemented in MATLAB based on the Robust Control Toolbox [17] and uses the technique of linear matrix inequalities (LMIs); see [18]. An illustrative example of the controller design for the well-known benchmark problem [5–8, 19–22] is given. This paper is an extended version of the results presented in [9, 20].

2. PRELIMINARIES

2.1. Description of the Plant and Controller

Consider a plant in physical variables described by the following equations:

$$\begin{aligned} L_1(p)\tilde{z}(t) &= L_2(p)u(t) + L_3(p)f(t), \\ y &= N\tilde{z}(t), \end{aligned} \tag{1}$$

where $\tilde{z}(t) \in R^l$ is the vector of physical variables of the plant (it contains coordinates, velocities, accelerations, currents, voltages, etc.); $u(t) \in R^m$ is the vector of control inputs; $y \in R^{m_2}$ is the vector of measured and, simultaneously, controlled variables; $f(t) \in R^{m_3}$ is the vector of unknown external disturbances described below. In addition, the matrix N is a known real matrix of dimensions $[m_2 \times l]$, and $L_1(p)$, $L_2(p)$, and $L_3(p)$ are polynomial matrices of dimensions $[l \times l]$, $[l \times m]$, and $[l \times m_3]$, respectively:

$$L_1(p) = \sum_{i=0}^{\alpha_1} L_1^{(i)} p^i, \quad L_2(p) = \sum_{j=0}^{\alpha_2} L_2^{(j)} p^j, \quad L_3(p) = \sum_{k=0}^{\alpha_3} L_3^{(k)} p^k,$$

where $L_1^{(i)}$, $L_2^{(j)}$, and $L_3^{(k)}$ are known real matrices of compatible dimensions; α_1, α_2, and α_3 are known numbers (α_2, $\alpha_3 < \alpha_1$), and $p = d/dt$ denotes the differentiation operator. By assumption, the plant (1) is stabilizable and detectable.

Let each component of the disturbance vector $f_i(t)$, $i = \overline{1, m_3}$, be a bounded polyharmonic function of the form

$$f_i(t) = \sum_{k=1}^{\infty} f_{ik} \sin(\omega_k t + \phi_{ik}), \quad i = \overline{1, m_3}, \tag{2}$$

where $f_{ik} > 0$, ϕ_{ik}, and ω_k, $i = \overline{1, m_3}$, $k = \overline{1, \infty}$, are the unknown amplitudes, phases, and frequencies of the disturbance, respectively. Suppose also that for each component, the amplitudes f_{ik} have a bounded sum:

$$\sum_{k=1}^{\infty} f_{ik} \leqslant f_i^*, \quad i = \overline{1, m_3}, \tag{3}$$

where f_i^* are given numbers.

In contrast to [6, 7], where the external disturbance was assumed to be bounded in the root-mean-square sense, the disturbances considered here may have an arbitrary number of harmonics with bounded absolute values.

By assumption, there exists a stabilizing controller for the plant (1) with nominal parameters:

$$u(t) = K(p) y(t), \tag{4}$$

where $K(p)$ is the transfer matrix of the controller containing proper transfer functions.

2.2. Robust Analysis

Suppose that the matrices $L_1^{(i)}$, $i = \overline{0, \alpha_1}$, and $L_2^{(j)}$, $j = \overline{0, \alpha_2}$, of the plant (1) contain n elements varying their values from known nominal ones $\lambda_1^0, \ldots, \lambda_n^0$. These variations can be described by functions of time

$$\lambda_i(t) = \lambda_i^0 + \triangle\lambda_i(t), \quad i = \overline{1, n},$$

where $\triangle\lambda_i(t)$ is the deviation of the ith element of the plant's matrix from its nominal value such that $\lambda_i(t) \in [\lambda_i^{\min}, \lambda_i^{\max}]$, and the interval bounds $\lambda_i^{\min}$ and $\lambda_i^{\max}$ are unknown.

System (1), (4) is asymptotically stable under the nominal values λ_i^0, $i = \overline{1, n}$, of the varying parameters $\lambda_i(t)$.

Problem 1. It is required to find admissible bounds $\lambda_i^{\min}, \lambda_i^{\max}$, $i = \overline{1, n}$, for which system (1), (4) is asymptotically stable.

The canonical (W, Λ, K)-form [5, 8, 9] is a special representation for the equations of system (1), (4):

$$\begin{aligned} \tilde{y} &= W_{11} \tilde{u} + W_{12} u, \quad \tilde{u} = \Lambda \tilde{y}, \\ y &= W_{21} \tilde{u} + W_{22} u, \quad u = K y, \end{aligned} \tag{5}$$

where $W_{ij}(p)$, $i, j = 1, 2$, are transfer functions not containing the varying parameters $\lambda_i(t)$; the signals $\tilde{u}(t)$ and $\tilde{y}(t)$ are functions from R^n, called the "*fictitious*" inputs and outputs, respectively, like the corresponding control loop; $\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$ is a diagonal matrix containing only the varying parameters; the control $u(t)$ and output $y(t)$ vectors have been described above.

As was shown in [6], the closed loop system (1), (4) can always be represented in the equivalent (W, Λ, K)-form.

System (5) with open loops of the inputs $\tilde{u}$ has the transfer matrix

$$W(p) = -\Lambda[W_{11} + W_{12}K(I - W_{22}K)^{-1}W_{21}] = -\Lambda\tilde{W}(p), \tag{6}$$

where I is an identity matrix of dimensions $(m_2 \times m_2)$.

An important feature of the transfer matrix (6) is that the varying parameters are contained only in the diagonal gain matrix Λ. Therefore, the circle criterion can be used to investigate the robust properties of the closed loop system.

Following the circle criterion, we consider the nonlinear control system [23]

$$\sigma(t) = -W(p)\xi(t), \quad \xi(t) = \varphi[\sigma(t), t], \tag{7}$$

where $W(p)$ is the linear part of the system (in the case under study, it represents the transfer matrix (6) of dimensions $[n \times n]$; $\varphi(\sigma, t) = [\varphi_1(\sigma_1, t), \ldots, \varphi_n(\sigma_n, t)]^T$ is the vector of nonlinear elements (generally, nonstationary) whose characteristics satisfy the inequalities

$$\alpha_i \leqslant \frac{\varphi_i(\sigma_i, t)}{\sigma_i} \leqslant \beta_i, \quad \varphi_i(0, t) = 0, \quad i = \overline{1, n}, \tag{8}$$

where $\alpha_i < 1$ and $\beta_i > 1$ determine the boundaries of the sectoral nonlinearity restricting the characteristic of the ith nonlinearity. System (7) with $\xi(t) = \sigma(t)$ (no nonlinearity) is asymptotically stable due to the controller design method in the case $\lambda_i = \lambda_i^0$.

With the current notations, we modify the circle criterion [9] as follows.

Theorem 1. *Let the transfer matrix of the linear part of the system satisfy the frequency matrix inequality*

$$[I + W(-j\omega)]^T[I + W(j\omega)] > R^2, \quad \omega \in [0, \infty), \tag{9}$$

where $R = \mathrm{diag}[r_1, \ldots, r_n]$ *is a diagonal matrix with elements* $0 < r_i \leqslant 1$, $i = \overline{1, n}$. *Then system* (7) *is absolutely stable under any nonlinearities from the class* (8) *such that*

$$\alpha_i = \frac{1}{1 + r_i}; \ \beta_i = \frac{1}{1 - r_i}; \quad i = \overline{1, n}. \tag{10}$$

Theorem 1 remains valid when replacing the nonlinear elements (8) with the linear nonstationary ones $\varphi_i[\sigma(t), t] = l_i(t)\sigma_i(t)$, $i = \overline{1, n}$:

$$\sigma(t) = -W(p)\xi(t), \quad \xi(t) = L(t)\sigma(t), \tag{11}$$

where $L(t) = \mathrm{diag}[l_1(t), \ldots, l_m(t)]$ is a diagonal matrix of nonstationary gains whose nominal values equal one. (In this case, system (11) is asymptotically stable for $\lambda_i = \lambda_i^0$, $i = \overline{1, n}$.)

For this case, Theorem 1 has an important consequence.

Corollary 1. *Let the transfer function* $W(p)$ *in* (11), *taken from* (6), *satisfy the frequency matrix inequality* (9). *Then system* (11) *is asymptotically stable in the large for any nonstationary gains within the bounds*

$$\frac{1}{1 + r_i} \leqslant l_i(t) \leqslant \frac{1}{1 - r_i}, \quad i = \overline{1, n}. \tag{12}$$

In contrast to [6, 7], these bounds are valid for the nonstationary case.

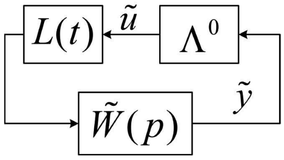

Fig. 1. The canonical $(W,\ \Lambda)$-form with nonstationary coefficients.

Problem 1 can be solved for an arbitrary set of physical parameters using the canonical form (5). The closed loop system (5) with the parameters $\lambda_i = \lambda_i^0$, $i = \overline{1,n}$, as before, is asymptotically stable by the controller design (4). By Corollary 1, in each control loop corresponding to $\tilde{u}_i$, it is possible to add nonstationary gains varying their values independently and arbitrarily within the bounds (12) so that the system will preserve its stability. The admissible bounds of these gains, $l_i(t),\ i = \overline{1,n}$, can be easily recalculated into the admissible bounds of the nonstationary physical parameters of the system. Figure 1 shows an explanatory block diagram in which $\tilde{W}$ is the same transfer matrix as in (6) and $\Lambda^0 = \mathrm{diag}[\lambda_1^0, \ldots, \lambda_n^0]$ is a diagonal matrix containing the nominal values of the varying parameters.

The robust stability criterion yielding the solution of Problem 1 was formulated in [9].

Theorem 2. *Let the transfer matrix* (6) *satisfy the frequency matrix inequality* (9)*. Then the system will preserve its stability under the following admissible bounds of the nonstationary independent variations of the parameters* $\lambda_i(t),\ i = \overline{1,n}$:

$$\frac{\lambda_i^0}{1 + r_i} \leqslant \lambda_i(t) \leqslant \frac{\lambda_i^0}{1 - r_i}, \quad i = \overline{1,n}, \tag{13}$$

where $\lambda_i^0 > 0,\ i = \overline{1,n}$, *are known nominal values of the varying parameters.*

The stationary bounds obtained in [5–7] involve the same (common) radius of stability margins. In contrast, the nonstationary bounds (13) of the varying parameters have the intervals determined by the individual radii of stability margins; see Theorem 4.

The bounds (13) are obviously valid also for the case of stationary variations of the parameters under study.

3. PROBLEM STATEMENT

Let us formulate two robust controller design problems: in the first, the external disturbances are absent (the stabilization problem); in the second, their effect is considered (the attenuation problem of external disturbances).

Problem 2 (robust stabilization). For the plant (1) it is required to find a controller (4) that will asymptotically stabilize the closed loop system (1), (4) under all nonstationary variations of the plant's physical parameters from their nominal values within given intervals:

$$\lambda_i^{\min} < \lambda_i(t) < \lambda_i^{\max}, \quad i = \overline{1,n},$$

where $\lambda_i^{\min},\ \lambda_i^{\max},\ i = \overline{1,n}$, are desired bounds.

The following problem includes external disturbances $f(t)$ from the class (2), (3).

The effect of such disturbances is considered by introducing system accuracy requirements in the form of steady-state value constraints on the errors of the controlled variables:

$$y_{i,st} = \lim_{t\to\infty} \sup |y_i(t)|, \quad i = \overline{1,m_2}.$$

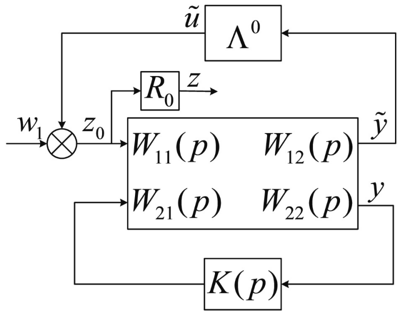

Fig. 2. The canonical $(W,\ \Lambda,\ K)$-form for Problem 2.

Problem 3 (the robust attenuation of external disturbances). For the plant (1) it is required to find a controller (4) that will asymptotically stabilize the closed loop system (1), (4) under all nonstationary variations of the plant's physical parameters from their nominal values within given intervals:

$$\lambda_i^{\min} < \lambda_i(t) < \lambda_i^{\max}, \quad i = \overline{1, n},$$

and will satisfy the accuracy requirements under unknown external disturbances from the class (2), (3) under the nominal values of the physical parameters:

$$y_{i,st} < \gamma y_i^*, \quad i = \overline{1, m_2}, \tag{14}$$

where $y_i^* > 0,\ i = \overline{1, m_2}$, are given numbers and $\gamma > 0$ is a given or minimized number.

4. SOLUTION OF THE PROBLEMS BASED ON H_∞ OPTIMIZATION

The solution of both problems is based on the technique of H_∞ optimization. In the standard formulation, it can be written as

$$\|T_{\overline{z}\overline{w}}\|_\infty < \gamma, \quad \overline{z} = T_{\overline{z}\overline{w}}(p)\overline{w}, \tag{15}$$

where $T_{\overline{z}\overline{w}}(p)$ is the transfer matrix of the closed loop system relating the extended input $\overline{w}$ to the extended output $\overline{z}$ and γ is a given or minimized number.

To apply this technique, we have to construct appropriate vectors $\overline{w}$ and $\overline{z}$ and matrix $T_{\overline{z}\overline{w}}(p)$ so that the solution of the corresponding H_∞ problem (15) also be the solution of the original Problems 2 and 3.

Problem (15) can be solved numerically with suitable standard software. For example, in this paper, we use *Robust Control Toolbox* of MATLAB: the H_∞ problems considered here may be singular and, consequently, require application of the LMI-based approach [24].

4.1. Solution of Problem 2

For Problem 2, the extended plant's equations have the form

$$\begin{gathered} \tilde{y} = W_{11}(p)z_0 + W_{12}(p)u, \quad \tilde{u} = \Lambda^0\tilde{y}, \\ y = W_{21}(p)z_0 + W_{22}(p)u, \quad u = K(p)y, \\ z = R_0 z_0, \quad z_0 = \tilde{u} + w_1, \quad R_0 = \operatorname{diag}[r_1^0, r_2^0, \ldots, r_n^0], \end{gathered} \tag{16}$$

where w_1 is the plant's fictitious input to make the closed loop system robust, $0 < r_i^0 \leqslant 1$, $i = \overline{1, n}$, are the desired values of the elements r_i^0 in the matrix R in (9). Figure 2 shows the block diagram of the (W, Λ, K)-form for this problem.

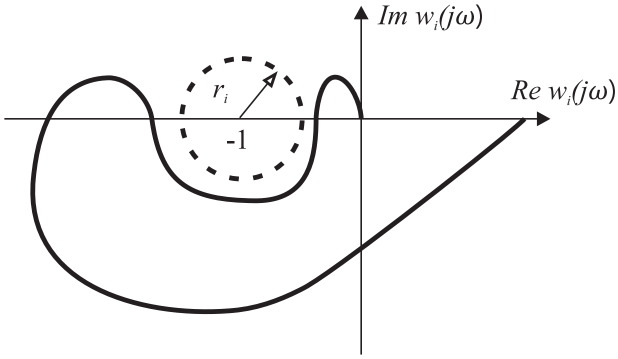

Fig. 3. Theorem 4 illustrated: r_i is the radius of the stability margins for the ith fictitious input $\tilde{u}_i$.

The corresponding extended input and output vectors and transfer matrix in (15) are written as

$$\overline{w} = w_1,\ \overline{z} = z,\ T_{\overline{z}\overline{w}} = R_0 T_{z_0 w_1}, \tag{17}$$

where $T_{z_0 w_1}$ is the transfer matrix of the closed loop system relating the vectors w_1 and z_0.

The solution of problem (15) for system (1), (4) possesses the following properties.

Theorem 3. *Let a controller $K(p)$ be found by solving the corresponding problem (15)–(17). It will stabilize system (1), (4) if*

$$\frac{\lambda_i^0}{1+r_i} \leqslant \lambda_i(t) \leqslant \frac{\lambda_i^0}{1-r_i}, \quad \lambda_i^0 > 0, \quad i = \overline{1,n},$$

where $r_i = r_i^0/\gamma$, $i = \overline{1,n}$, r_i^0 are the given values of the elements of the diagonal matrix R_0, and γ is the value realized when solving problem (15).

The proofs of this and subsequent assertions are provided in the Appendix.

Now we describe the frequency properties of the designed systems. Let $w_i(p)$ be the transfer function of the open loop system (5) relative to the ith fictitious input $\tilde{u}_i$ (i.e., the ith varying parameter λ_i). Then the following result is true.

Theorem 4. *Assume that the frequency matrix inequality (9) holds. Then the Nyquist plot of system (5) with an open loop of the ith fictitious input $\tilde{u}_i$ does not touch the circle of radius r_i centered at the critical point $(-1, j0)$ on the complex plane $w_i(j\omega)$.*

Theorem 4 gives a physically important interpretation of the frequency matrix inequality (9), see Fig. 3. That is, the diagonal elements of the matrix $R = R_0/\gamma$ determine the radius r_i of the stability margins of the system with an open loop of the ith input $\tilde{u}_i$. In addition, the parameter λ_i is the factor of the transfer function $w_i(p)$, and the designer-specified diagonal elements r_i^0 of the matrix R_0 determine the desired radii of stability margins for the ith input $\tilde{u}_i$.

4.2. Solution of Problem 3

Problem 3 is also solved using the standard H_∞ problem (15) in which the extended plant's equations are described similar to [6, 7]:

$$\begin{gathered} \tilde{y} = W_{11}(p)z_1 + W_{12}(p)u + W_{13}(p)f, \quad \tilde{u} = \Lambda^0\tilde{y}, \\ y = W_{21}(p)z_1 + W_{22}(p)u + W_{23}(p)f, \quad u = K(p)y, \\ z_1 = R_0(\tilde{u} + w_1), \quad z_2 = Q^{1/2}y, \end{gathered} \tag{18}$$

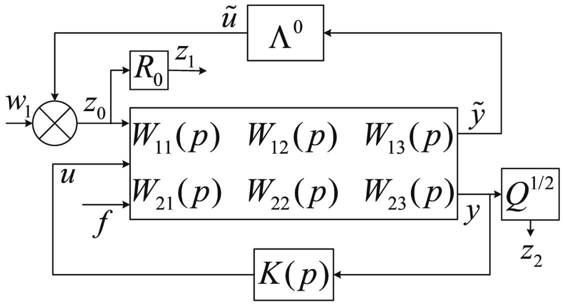

Fig. 4. The canonical $(W,\ \Lambda,\ K)$-form for Problem 3.

where $Q = \text{diag}[q_1, \ldots, q_{m_2}]$ is a weight matrix with elements $q_i > 0$, $i = \overline{1, m_2}$, assigned to satisfy the system accuracy requirements (14). Figure 4 shows the block diagram of the (W, Λ, K)-form for this problem.

In view of the block diagram in Fig. 4 and the corresponding equations (18), we obtain the vectors

$$\overline{w}^T = [w_1^T \quad f^T], \quad \overline{z}^T = [z_1^T \quad z_2^T] = [(w_1^T + \tilde{u}^T)R_0 \quad y^T Q^{1/2}]$$

and the following closed-loop system matrix $T_{\overline{zw}}$:

$$\overline{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = T_{\overline{zw}}(p)\overline{w} = \begin{bmatrix} R_0 T_{z_0 w_1} & R_0 T_{z_0 f} \\ Q^{1/2} T_{y w_1} & Q^{1/2} T_{y f} \end{bmatrix} \times \begin{bmatrix} w_1 \\ f \end{bmatrix}, \tag{19}$$

where $T_{z_0 w_1}$, $T_{z_0 f}$, $T_{y w_1}$, and T_{yf} are the transfer matrices of the closed loop system relating w_1 to z_0, f to z_0, w_1 to y, and f to y, respectively.

Theorem 5. *Let problem* (15), (19) *be solved with the elements of the diagonal matrix* Q *assigned by*

$$q_i = \left(\sum_{j=1}^{m_3} f_j^* \right)^2 / (y_i^*)^2, \quad i = \overline{1, m_2}.$$

Then the controller $K(p)$ *asymptotically stabilizes system* (1), (4) *and ensures the steady-state errors within the intervals*

$$\begin{gathered} \frac{\lambda_i^0}{1 + r_i} \leqslant \lambda_i(t) \leqslant \frac{\lambda_i^0}{1 - r_i}, \quad \lambda_i^0 > 0, \quad i = \overline{1, n}, \\ y_{j,st} < \gamma y_j^*, \quad j = \overline{1, m_2}, \end{gathered} \tag{20}$$

where $r_i, i = \overline{1, n}$, *and* γ *are the same variables as in Theorem 3 and* $y_j^* > 0$, $j = \overline{1, m_2}$, *are the desired control errors.*

According to the proof of Theorem 5, the closed loop system (1), (4) or (5) preserves stability under all nonstationary variations of the plant's physical parameters within the intervals specified by the first inequality in (20). In addition, the guaranteed r_i and desired r_i^0 radii of the stability margins and the value γ realized when solving problem (15), (19) are related by $r_i = r_i^0/\gamma$, $i = \overline{1, n}$.

Note that when assigning the weight matrix Q as prescribed by Theorem 5, the second inequality in (A.4) (see the Appendix) fulfills the accuracy requirements (the second inequality in (20)) only under the nominal values of the varying parameters $\lambda_i = \lambda_i^0$, $i = \overline{1, n}$.

5. EXAMPLE

As an illustrative example, we take the well-known benchmark problem of a two-mass system [21, 22]. The system consists of two bodies connected by a spring and is described by the equations

$$\dot{x}_1 = x_3, \quad \dot{x}_2 = x_4, \quad \dot{x}_3 = -qx_1 + qx_2 + u + f, \quad \dot{x}_4 = qx_1 - qx_2, \quad y = x_2, \tag{21}$$

where q is the stiffness of the spring, which varies in some unknown interval. The nominal value q is chosen equal to $q_0 = 0.8$. The external disturbance signal f is applied at the same point as the control input. Figure 5 shows the structure of the plant (21) closed by the desired controller $K(p)$.

Let us explain the physical motivation for considering the nonstationary values of the spring stiffness $q(t)$: Hooke's linear law is valid only under small deviations from the equilibrium (compressing or stretching the spring). For large deviations from the equilibrium (significant stretchings or compressions), this law becomes nonlinear, and it can be represented as a linear nonstationary law with nonstationary variations of the spring stiffness [25].

In Problem 3, it is necessary to provide a given accuracy for the controlled and, simultaneously, measured variable y of the closed loop system and maximize the stability margins in the fictitious control loop containing the varying parameter q.

System (21) has been written in the (W, Λ, K)-form (18); for details, see [6, 7]. Therefore, we proceed directly to formulating the H_∞ problem (15) for the particular system under consideration. Problem (15) is solved using standard MATLAB software based on LMIs. For this purpose, we represent the plant equations in the generalized state-space form

$$\dot{x} = Ax + B_1\overline{w} + B_2u; \quad \overline{z} = C_1x + D_{11}\overline{w} + D_{12}u;$$
$$y = C_2x + D_{21}\overline{w} + D_{22}u.$$

The generalized plant's matrices are

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -q & q & 0 & 0 \\ q & -q & 0 & 0 \end{bmatrix}; \; B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 1 \\ -1 & 0 \end{bmatrix};$$

$$B_2 = [0\;0\;1\;0]^T;$$

$$C_1 = \begin{bmatrix} R_0 \times (-q \;\; q \;\; 0 \;\; 0) \\ Q^{1/2} \times (0 \;\; 1 \;\; 0 \;\; 0) \end{bmatrix}; \; D_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix};$$

$$C_2 = [0\;1\;0\;0]; \; D_{12} = [0\;0]^T; \; D_{21} = [0\;0]; \; D_{22} = 0.$$

The following parameters were used in the numerical experiment: the desired control error $y^* = 0.5$ and the bound $f^* = 1$ for the external disturbance. Then Theorem 5 yields the corresponding weight coefficient:

$$Q = (f^*)^2/(y^*)^2 = 4, \quad Q^{1/2} = 2.$$

The second design parameter (the desired radius of stability margins) was chosen as $R_0 = 0.9$.

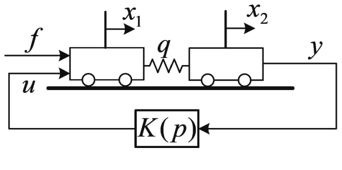

Fig. 5. Two-mass-spring system.

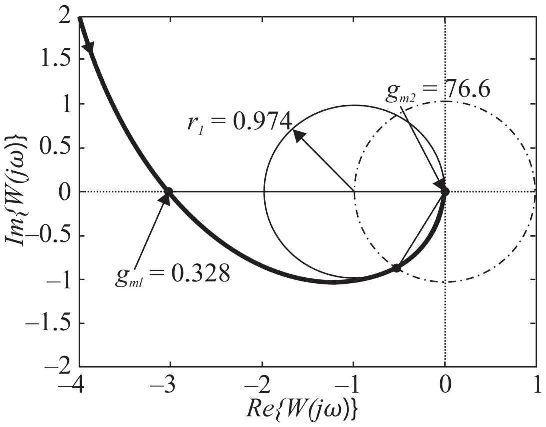

Fig. 6. The Nyquist plot of the open loop system.

The controller design by the `hinflmi` function of Robust Control Toolbox yields

$$K(p) = -\frac{1.045 \times 10^{12}(p + 0.5409)(p^2 + 0.713p + 1.049)}{(p^2 + 256.8p + 5.289 \times 10^4)(p^2 + 202.9p + 4.86 \times 10^6)}. \tag{22}$$

The optimal value $\gamma = 1.01895$ gives the radius $R = R_0/\gamma = 0.883$ of stability margins, with which the system preserves stability under the nonstationary variations of the parameter q (from the nominal value $q_0 = 0.8$) within the interval

$$0.425 \leqslant q(t) \leqslant 6.853.$$

This corresponds to $q^{\min} = q_0/(1 + R) = 0.425$ and $q^{\max} = q_0/(1 - R) = 6.853$. The numerical bounds of the admissible variations of the nonstationary parameter $q(t)$ under which the system remains stable can be further refined by using Theorem 2, which is based on the circle criterion (also, see [23]). The true value of the parameter $R = r_1$ in (9) is 0.974; it was found from the Nyquist plot of the corresponding transfer function of the open loop system (6) for $\Lambda = q_0$ (Fig. 6). The bounds obtained from the Nyquist plot are much wider than the guaranteed ones and are

$$0.406 \leqslant q(t) \leqslant 30.995.$$

In the case of the stationary uncertainty of the parameter q, we get even wider robust stability bounds:

$$0.263 \leqslant q \leqslant 61.297.$$

These bounds were also obtained from the Nyquist plot (Fig. 6). Here, the point g_{m1} (g_{m2}) shows how many times the loop gain can be decreased (increased, respectively) without stability loss. The varying parameter q is the factor of the transfer function of the open loop, which explains the derivation of these bounds.

The resulting bounds are much wider than the well-known in the literature. A comparative analysis of the most successful controller design approaches for the plant (21) was carried out in [19].

The classical stability margins (the phase and amplitude margins) are found for the open control loop for the real input u (or output y). In this example, they equal $L = 35.4$ dB and $\phi_{\text{mar}} = 64.5°$.

Figure 7 shows the transient response of the closed loop system (21), (22) under $f = 1$. Clearly, the steady-state value $y_{st} = 0.434$ of the controlled variable satisfies the accuracy requirements $y_{st} < y^* = 0.5$. The realized value of the control error insignificantly differs from the desired one,

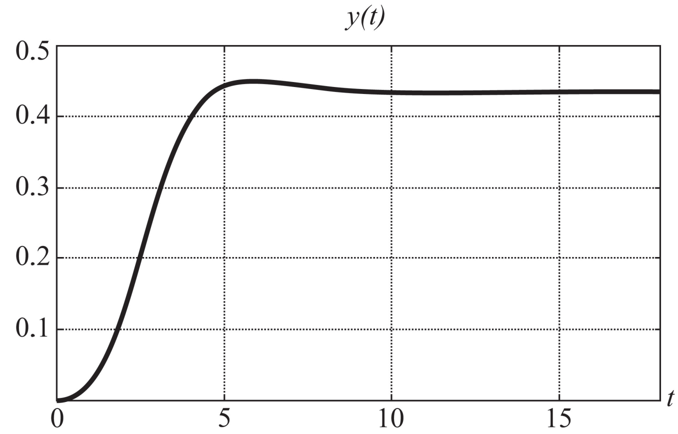

Fig. 7. The transient response of the closed loop system.

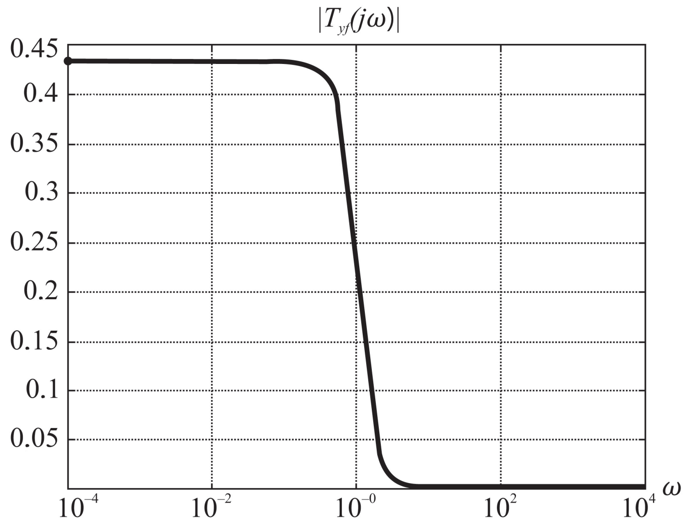

Fig. 8. The amplitude-frequency response of the closed loop system.

which indicates a low degree of sufficiency of the design method in terms of this performance index. Figure 8 presents the amplitude-frequency response of the transfer function relating f to y, which is a monotonically decreasing curve. Hence, the step external disturbance is the worst for system (21), (22).

In the example, the disturbance f is applied together with the control input to the first body, and the accuracy requirements are fulfilled. Note that they will also hold when applying f to the second body.

6. CONCLUSIONS

This paper has proposed a control design method that can be used in real engineering problems, as it possesses several advantages:
(a) The practically important class (2), (3) of external disturbances is rather wide: it covers elementwise bounded functions of time, $|f_i(t)| \leqslant f_i^*,\ i = \overline{1, m_3}$, which are, in particular, continuous and piecewise differentiable. Hence, they can be represented by an absolutely convergent Fourier series (2) (if the frequencies are multiples) [10]. (Such disturbances are only physically possible in engineering practice.)

(b) The method considers nonstationary deviations of the plant's physical parameters.
(c) The control design procedure is reduced to solving the standard H_∞ optimization problem.
(d) The design method is noniterative, and the resulting controller has an order not exceeding that of the plant.

At the same time, according to the studies [6, 8, 19], even despite the significant bounds of the admissible deviations of the physical parameters from the nominal ones (under which the closed loop system remains stable), the radius of stability margins at the plant's physical input or output may be very small, which makes the resulting controller unusable in practice [10]. Therefore, the challenge is to improve this design method toward the additional consideration of the requirements for the radius of stability margins at the plant's physical input or output [8], even though in the illustrative example, the stability margins at the open loop points have turned out pretty good.

In addition, it is necessary to consider system time response requirements during controller design, even though in the illustrative example, the performance of the closed loop system (the settling time) has turned out small enough, $t_p \approx 10$ s.

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APPENDIX

Proof of Theorem 3. From (15) and (17) it follows that

$$T^T_{z_0w_1}(-j\omega)R_0^2T_{z_0w_1}(j\omega) < \gamma^2 I, \quad \omega \in [0,\infty). \tag{A.1}$$

Since $T_{z_0w_1} = [I + W(j\omega)]^{-1}$ (see [5–7]), we obtain

$$[I + W(-j\omega)]^T[I + W(j\omega)] > R_0^2/\gamma^2, \quad \omega \in [0,\infty), \tag{A.2}$$

which coincides with (9), where $R^2 = R_0^2/\gamma^2$. By Theorem 2, the admissible bounds are the same as in Theorem 3.

As $\omega \to \infty$ condition (A.2) implies $I > R_0/\gamma$. Hence, the realized value γ satisfies the inequalities

$$\gamma > r_i^0, \quad i = \overline{1,n}.$$

Proof of Theorem 4. Let $t_i(p)$ be the transfer function of the closed loop system relating the ith component of the vector w_1 to the ith component of the vector z_0. Due to the diagonal structure of the matrix R_0, the transfer function relating w_{1i} to z_i is $r_i^0 t_i(p)$. In turn, the functions $t_i(p)$ and $w_i(p)$ have an analog of the classical relation [10]:

$$t_i(p) = 1/[1 + w_i(p)].$$

Thus, the transfer function relating w_{1i} to z_i is $r_i^0/[1 + w_i(p)]$; on the other hand, it represents the ith diagonal element of the transfer matrix $R_0T_{z_0w_1}$, which satisfies inequality (15) by (17). Consequently, any of its elements satisfies an analogous condition, and then $\|r_i^0/[1+w_i(p)]\|_\infty < \gamma$, which can be equivalently written as

$$[1 + w_i(-j\omega)][1 + w_i(j\omega)] > (r_i^0/\gamma)^2 = r_i^2, \quad \omega \in [0,\ \infty). \tag{A.3}$$

Proof of Theorem 5. Suppose that problem (15) has been solved for the matrix $T_{\overline{zw}}(p)$ (19). Then the corresponding inequalities are valid for each separate block of the transfer matrix. Consider only the diagonal blocks:

$$\|R_0 T_{z_0 w_1}\|_\infty < \gamma, \quad \|Q^{1/2} T_{yw}\|_\infty < \gamma. \tag{A.4}$$

The first inequality can be represented as (A.1), and the same considerations as in Theorem 3 are applicable here. As a result, the first inequality in (20) holds.

The second block inequality in (A.4) ensures the accuracy requirements (14). It can be equivalently written as

$$T_{yw}^T(-j\omega) Q T_{yw}(j\omega) < \gamma^2 I.$$

According to [10], with the latter inequality being true, the steady-state values of the controlled variables satisfy the inequalities

$$q_i y_{i,st}^2 < \gamma^2 \left(\sum_{j=1}^{m_3} f_j^* \right)^2, \quad i = \overline{1, m_2}, \tag{A.5}$$

under any disturbance from the class (2), (3).

Let us assign the weights q_i, $i = \overline{1, m_2}$, as prescribed by Theorem 5 and substitute them into (A.5). After straightforward transformations, we finally arrive at the desired inequality (14). Thus, both inequalities in (20) hold, and the proof of this theorem is complete.

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